

ON THE THIN BOUNDARY OF THE FAT ATTRACTOR

ARTUR O. LOPES AND ELISMAR R. OLIVEIRA

ABSTRACT. Following M. Tsujii, for, $0 < \lambda < 1$, consider the transformation $T(x) = dx \pmod{1}$ on the circle S^1 , and, the map $F(x, s) = (T(x), \lambda s + A(x))$, $(x, s) \in S^1 \times \mathbb{R}$.

This transformation F (which is not bijective) defines an attractor set and its dynamics is similar to the solenoid. For each value λ the attractor is the set of points which have a bounded set of pre-images (of arbitrary order). For each λ one can consider λ -maximizing probabilities and this is an important ingredient in the proof of our main results.

We denote $\mathcal{B} = \mathcal{B}_\lambda$ the upper boundary of the attractor. We are interested in probabilities on \mathcal{B}_λ , and, what happen in the limit when $\lambda \rightarrow 1$. From another point of view, the topological one, it is natural to try to understand the regularity of this boundary. We use in an essential way several results which are taken from Ergodic Transport Theory and also an involution kernel.

We also address the analysis of following conjecture which were proposed by R. Bamón, J. Kiwi, J. Rivera-Letelier and R. Urzúa: for any fixed λ , generically C^1 on the potential A , the upper boundary \mathcal{B}_λ is formed by a finite number of pieces of smooth unstable manifolds of periodic orbits for F . We prove the conjecture in the sense that for a generic (twist) C^1 potential there is an $\epsilon > 0$ such that in the interval $(1 - \epsilon, 1)$ the λ -maximizing probability is a periodic orbit. As we will see this proves the conjecture for a generic twist A for $\lambda \in (1 - \epsilon, 1)$. We show in the last section an specific analysis of the conjecture for any $\lambda \in (0, 1]$ for the smooth case when $A(x)$ is quadratic.

1. INTRODUCTION

Consider, $0 < \lambda < 1$, the transformation $T(x) = dx \pmod{1}$, $d \in \mathbb{N}$, and, the map $F(x, s) = (T(x), \lambda s + A(x))$, $(x, s) \in S^1 \times \mathbb{R}$. We denote by τ_i , $i = 1, 2, \dots, d$, the inverse branches of T .

Note that $F^n(x, s) = (T^n(x), \lambda^n s + [\lambda^{n-1}A(x) + \lambda^{n-2}A(T(x)) + \dots + \lambda A(T^{n-2}(x)) + A(T^{n-1}(x))])$.

Here we use sometimes the natural identification of S^1 with the interval $[0, 1)$.

We consider here A sometimes of class C^1 (as in [35]), piecewise C^1 , or, Lipschitz (as in [6]).

Most of the results presented here also applied to the case of the shift (instead of the T above) on the Bernoulli space and a Holder or Lipschitz potential.

This transformation F (is not bijective) defines an attractor set and its dynamics is similar to the one of the solenoid. For each λ the attractor is the set of points which have a bounded set of pre-images (of arbitrary order) [6]. We denote $\mathcal{B} = \mathcal{B}_\lambda$ the upper boundary of the attractor. We are interested in probabilities on \mathcal{B}_λ and what happen in the limit when $\lambda \rightarrow 1$. The analysis of the lower boundary is similar to the case of the upper boundary and will not be considered here.

The study of the dimension of the boundary of strange attractors is a topic of great relevance in non-linear physics [32] [20]. The papers [21] and [22] discuss somehow related questions. We want to analyze a case where this boundary is piecewise smooth and one-dimensional.

In figure 1 we show the image of the fat attractor for the case of $A(x) = (x - 0.5)^2$, $\lambda = 0.51$, and $d = 2$.

One of our main motivations for this work is the following conjecture (see [6]): for any fixed λ , generically C^1 on the potential A , the upper boundary \mathcal{B}_λ is formed by a finite number of pieces of unstable manifolds of periodic orbits for F .

The twist property for a potential A is defined in the last section. We will prove this conjecture in the following sense: for a generic twist potential A and λ close to one the upper boundary \mathcal{B}_λ is formed by a finite number of pieces of unstable manifolds of periodic orbits for F . This follows from section 2, the last section and Theorem 4. In the last section of the paper, we will develop techniques that will help to get a better understanding of the difficulties for analyzing the case of any $\lambda \in (0, 1]$. The mechanism behind the dynamics of such problem can be better understood under the perspective of Ergodic Transport Theory (see [29] [30] [28] [31]). This is the purpose of section 2. Finally, we apply all this to the case when A is quadratic in the last section. We will assume from now on that A is at least of Holder class.

We point out that the twist condition is an open property in the C^2 topology. The main problem we analyze here could also be expressed in the C^2 topology.

Definition 1. *It is known (see [6]) that the upper boundary of the attractor is the graph of a Lipschitz function $b : S^1 \rightarrow \mathbb{R}$, where $b = b_\lambda$ satisfies*

$$b(x) = \max_{T(y)=x} \{\lambda b(y) + A(y)\}.$$

We call b_λ the λ -calibrated subaction.

A geometric description of the above equation is presented in figure 2.

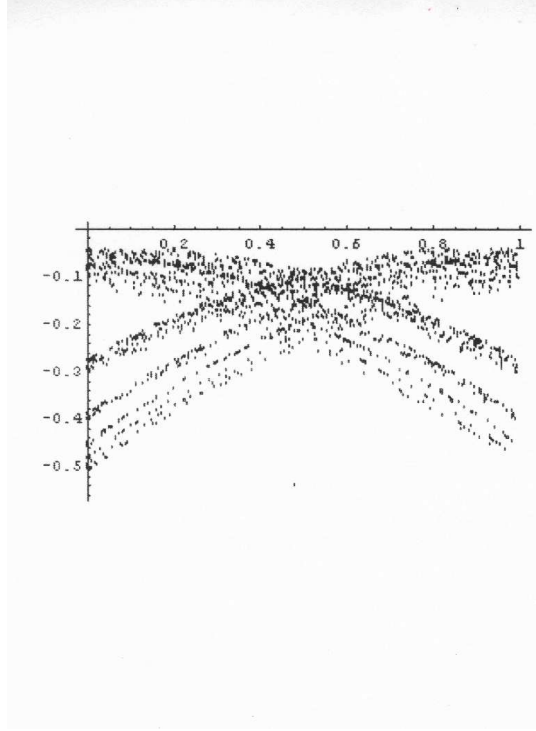


Fig 1) The fat attractor for the case of $A(x) = (x - 0.5)^2$, $\lambda = 0.51$, and $d = 2$.

The function b above exist and it is unique (see [11]). The existence of such b when A is Holder is also presented in the survey paper [27]. Note that if b is the λ -calibrated subaction for A , then, $b + \frac{g}{\lambda}$ is the λ -calibrated subaction for $A + \frac{g \circ T}{\lambda} - g$.

Given x , if i_0 is such that $b(x) = \lambda b(\tau_{i_0}(x)) + A(\tau_{i_0}(x))$, we say $\tau_{i_0}(x)$ realizes $b(x)$ (or, realizes x). We can also say that i_0 is a symbol which realizes $b(x)$.

One can show that for $d = 2$, for any Holder A , there exist x such that $b(x)$ has two different $\tau_{i_0}(x)$ realizers. In this way realizers are not always unique.

For fixed $x_0 \in S^1$ consider x_1 such that $b(x_0) = \lambda b(x_1) + A(x_1)$, and $T(x_1) = x_0$.

Then, there exist a realizer i_0 such that $\tau_{i_0}(x_0) = x_1$.

Now take x_2 such that $b(x_1) = \lambda b(x_2) + A(x_2)$ and $T(x_2) = x_1$.

In the same way as before, there exist i_1 such that $\tau_{i_1}(x_1) = x_2$.

In this way get by induction a sequence $x_k \in S^1$ such that $T(x_k) = x_{k-1}$.

This also defines an element $\bar{a} = a(x_0) = (i_0, i_1, \dots, i_n, \dots) \in \Sigma = \{1, \dots, d\}^{\mathbb{N}}$, where $\tau_{i_k}(x_k) = x_{k+1}$. This w depends of the choice of realizers we choose in the sequence of preimages. We say $(x_0, a(x_0)) \in S^1 \times \{1, 2, \dots, d\}^{\mathbb{N}}$ is an optimal pair. Note that for each $x_0 \in S^1$ there exist at least one optimal pair. For each x_0 we consider a fixed choice $a(x_0)$, and, the corresponding sequence $x_k \in S^1$, $k \in \mathbb{N}$.

Consider the probability $\mu_n = \sum_{j=0}^{n-1} \frac{1}{n} \delta_{x_n}$ and μ_λ any weak limit of a convergent subsequence μ_{n_k} , $k \rightarrow \infty$.

The probability μ_λ on S^1 is T invariant and satisfies

$$\int (b(T(x)) - \lambda b(x) - A(x)) d\mu_\lambda = 0.$$

Note that

$$b(T(z)) - \lambda b(z) - A(z) \geq 0$$

for all $z \in S^1$.

In this way for z in the support of μ_λ we get the λ -**cohomological** equation $b(T(z)) - \lambda b(z) - A(z) = 0$.

Therefore, μ_λ is maximizing for the potential $A(z) - b(T(z)) + \lambda b(z)$.

Definition 2. Any probability which maximizes $A(z) - b(T(z)) + \lambda b(z)$, where b is the λ -calibrated subaction, will be called a λ -maximizing probability for A .

If $b_\lambda^\& = b_\lambda - \max b_\lambda$, then, of course, μ_λ is maximizing for the potential $A(x) - b_\lambda^\&(T(z)) + \lambda b_\lambda^\&(z)$. General references for maximizing probabilities are [5] [23] [27] [12] [10][11] [26] [18] [8] [9] [33] [27] .

As we are maximizing among invariant probabilities we can also say that μ_λ is maximizing for the potential

$$A(z) + (\lambda - 1)b(z) = A(z) - b(T(z)) + \lambda b(z) + [b(T(z)) - b(z)].$$

One of our main results is Theorem 4 which claims that for a generic Lipchitz potential which satisfies the twist condition (to be defined later) there exists an $\epsilon > 0$ such that for $\lambda \in (1 - \epsilon, 1]$ the λ -maximizing probability is a periodic orbit. This will follow from a recent result by G. Contreras (see [14]), proposition 18 and Theorem 4 which uses the property of continuous varying support property (see [12]). As we will see in next section this is associated, via the λ -involution kernel, to the possibility of controlling the associated finite set of Lipchitz pieces of the boundary of the attractor. This pieces (in a finite number) will be of class C^1 in the C^1 setting and class C^2 in the C^2 setting.

We point out that in order to obtain examples where one can determine the maximizing probability or calibrated subactions for a given potential A it is necessary to know at least the exact value the maximal value $m(A)$ of the possible integrals of A among the invariant probabilities. In the general case this is not an easy task and therefore any method of approximation of this maximal value or associated subaction is helpful. The discounted method provides approximations b_λ of the calibrated subaction for A via the Banach fixed point theorem, that is, via a contraction in the set of continuous functions. It is not necessary to know the value $m(A)$. However, when λ becomes close to the value 1 this contraction becomes weaker and weaker. Theorem 4 claims that for a generic potential A the maximizing measure for A is attained by a λ -maximizing probability for λ in an interval of the form $[1 - \epsilon, 1]$. Thanks to that one can apply our reasoning for a λ bounded away from 1.

The same result is true for a C^1 potential A and the C^1 topology.

Definition 3. We denote by R the function $R = A - b \circ T + \lambda b$ and call it the rate function.

For fixed λ , by the fiber contraction theorem [34] (section 5.12 page 202 and section 11.1 page 433) we get that the λ -calibrated subaction $b = b_{\lambda,A} = b_A$ is a continuous function of A . Moreover, the function $b = b_{\lambda,A}$ is a continuous function of A and λ .

As usual we denote $m(A) = \max\{\int A d\rho \mid \text{where } \rho \text{ is } T \text{ invariant}\}$. We call maximizing probability for A any ρ which attains the supremum. we denote any of these by μ_A .

A continuous function $U : S^1 \rightarrow \mathbb{R}$ is called a calibrated subaction for $A : S^1 \rightarrow \mathbb{R}$, if, for any $y \in S^1$, we have

$$(1) \quad U(y) = \max_{T(x)=y} [A(x) + U(x) - m(A)].$$

It is known (see [27] and [4] for references) that b_λ is equicontinuous in λ , and, any convergent subsequence, $\lambda_n \rightarrow 1$, satisfies $b_{\lambda_n}^\& \rightarrow U$, where U is a calibrated subaction for A (see [17] for definition). Assuming the maximizing probability for A is **unique** (a generic property according to [12]), it is known (see [17] section 4), that when $\lambda \rightarrow 1$, we get that $b_\lambda^\& \rightarrow U$, where U is a (the) calibrated subaction for A . In this way we can say that the family $b_\lambda^\& \rightarrow U$ selects the calibrated subaction U .

In this way, under this hypothesis, when $\lambda \rightarrow 1$, we get that \mathcal{B}_λ (after the subtraction of $\max b_\lambda$) converges to the graph of the calibrated subaction for A . I

Note that in any case it is true the relation: for any z

$$b^\&(T(z)) - \lambda b^\&(z) + (1 - \lambda)(\max b) - A(z) \geq 0.$$

It is known (see [27]) that $(1 - \lambda)(\max b) \rightarrow m(A)$.

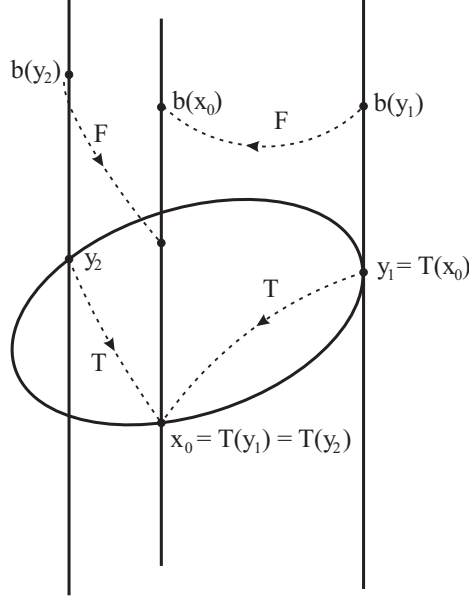


Fig 2).

Theorem 4. *If ν is a weak limit of a converging subsequence $\mu_{\lambda_n} \rightarrow \nu$, $\lambda_n \rightarrow 1$, then, ν is a maximizing probability for A . For a generic Lipchitz potential A there exist an ϵ , such that for $\lambda \in (1 - \epsilon, 1]$, the λ -maximizing probability is a periodic orbit and equal its limit. If the potential A is of class C^1 the same is true on the C^1 topology.*

Proof: Consider a subsequence $\mu_{\lambda_n} \rightarrow \nu$, $\lambda_n \rightarrow 1$. Such ν is clearly invariant. Suppose by contradiction that for some $\epsilon > 0$ there exists an invariant μ such that

$$\int (A - U \circ T + U) d\nu + \epsilon < \int (A - U \circ T + U) d\mu,$$

then, for any n large enough we have

$$\int (A - b_{\lambda_n} \circ T + \lambda_n b_{\lambda_n}) d\mu_{\lambda_n} + \epsilon/2 < \int (A - b_{\lambda_n} \circ T + \lambda_n b_{\lambda_n}) d\mu,$$

and, we reach a contradiction.

Now, for a generic potential it is known that the maximizing probability for A is a unique periodic orbit (see [14]). Therefore, $\mu_\lambda \rightarrow \nu$, when $\lambda \rightarrow 1$.

From the continuous varying support (see [12]) if $\mu_\lambda \rightarrow \nu$ and ν is periodic orbit, then, there exist an $\epsilon > 0$ such that for $\lambda \in (1 - \epsilon, 1]$ the probability $\mu_\lambda = \nu$.

If the potential A is of class C^1 one can get a Lipchitz subaction and perturb in the same way as in [14] in order to get a close by potential with support in a unique periodic orbit. This potential can be approximated in the C^1 topology and again by the continuous varying support we get a close by C^1 potential with support in a periodic orbit. Then, the same formalism as above can be applied. \square

If $\mu_\lambda \rightarrow \mu_A$, when $\lambda \rightarrow 1$, we say that μ_λ selects the maximizing probability μ_A . In the present case for a generic A there is a selection via the discounted method.

An interesting example is $A(x) = -(x - 0.5)^2$ and $T(x) = 2x \pmod{1}$, which has a unique maximizing probability which is the one with support in the periodic orbit of period 2 [24] [25]. Therefore, the corresponding λ -maximizing probability μ_λ converges to this one. This example will be carefully analyzed in the last section of the paper.

2. DUALITY AND OPTIMAL TRANSPORT

Following the notation of the beginning of last section we point out that: given $x = x_0$, there exists a sequence $x_k \in S^1$, $k \in \mathbb{N}$, such that

$$b(x_{k-1}) - \lambda b(\tau_{i_k}(x_k)) - A(\tau_{i_k}(x_k)) = 0.$$

One can consider the probability $m_n = \sum_{j=0}^{n-1} \frac{1}{n} \delta_{\sigma^j(\bar{a})}$, where σ is the shift, and, $\bar{a} = a(x_0)$ is optimal for x_0 . We define the probability μ_λ^* in $\{1, 2, \dots, d\}^{\mathbb{N}}$, as any weak limit of a convergent subsequence m_{n_k} , $k \rightarrow \infty$ (which will be σ invariant).

Definition 5. We call μ_λ^* a λ -dual probability for A .

Consider (as Tsujii in [35]) the function $S : (S^1, \Omega) \rightarrow \mathbb{R}$, where $\Omega = \{1, 2, \dots, d\}^{\mathbb{N}}$, given by

$$S_{\lambda, A}(x, a) = S(x, a) = \sum_{k=0}^{\infty} \lambda^k A((\tau_{a_k} \circ \tau_{a_{k-1}} \circ \dots \circ \tau_{a_0})(x)),$$

and, $a = (a_0, a_1, a_2, \dots)$.

Note that for a fixed a the function $S(x, a)$ is not periodic on x .

For each fixed a the function on $x \rightarrow S(x, a)$ is C^2 , if, A is C^2 (page 1014 [35]), and, $1 > \lambda > \frac{1}{d}$. Note also that for λ and a fixed the function $S_{\lambda, A}(\cdot, a)$ is linear in A .

All x has a corresponding $a = a(x)$ such that $b(x) = S(x, a)$. Indeed, for the given x take i_0 such that

$$b(x) = \lambda b(\tau_{i_0}(x)) + A(\tau_{i_0}(x)),$$

then, take i_1 such that

$$b(\tau_{i_0}(x)) = \lambda u((\tau_{i_1} \circ \tau_{i_0})(x)) + A((\tau_{i_1} \circ \tau_{i_0}(x))),$$

and so on. In this way we get $a = (i_0, i_1, i_2, \dots)$. This a is not necessarily unique.

Note that

$$\begin{aligned} b(x) &= \lambda [\lambda u((\tau_{i_1} \circ \tau_{i_0})(x)) + A((\tau_{i_1} \circ \tau_{i_0}(x)))] + A(\tau_{i_0}(x)) = \\ &= \lambda^2 u((\tau_{i_1} \circ \tau_{i_0})(x)) + \lambda A((\tau_{i_1} \circ \tau_{i_0}(x))) + A(\tau_{i_0}(x)) = \\ &= \lambda^n u((\tau_{i_{n-1}} \circ \dots \circ \tau_{i_1} \circ \tau_{i_0})(x)) + \\ &+ \lambda^{n-1} A((\tau_{i_{n-1}} \circ \dots \circ \tau_{i_1} \circ \tau_{i_0}(x))) + \dots + \lambda A((\tau_{i_1} \circ \tau_{i_0}(x))) + A(\tau_{i_0}(x)). \end{aligned}$$

Taking the limit when $n \rightarrow \infty$ we get $b(x) = S(x, a)$.

Note from the construction that for any other $c \in \{1, 2, \dots, d\}^{\mathbb{N}}$ we get that $b(x) \geq S(x, c)$. Indeed, consider

$$\begin{aligned} z(x) &= \limsup_{n \in \mathbb{N}} \{ \lambda^{n-1} A((\tau_{i_{n-1}} \circ \dots \circ \tau_{i_1} \circ \tau_{i_0})(x)) + \dots \\ &+ \lambda A((\tau_{i_1} \circ \tau_{i_0}(x))) + A(\tau_{i_0}(x)) \mid (i_0, i_1, \dots, i_{n-1}) \in \{1, 2, \dots, d\}^n \}, \end{aligned}$$

and, the operator

$$\mathcal{L}_\lambda(v)(x) = \sup_{i=1,2,\dots,d} [A(\tau_i(x)) + \lambda v(\tau_i(x))].$$

Then, $\mathcal{L}_\lambda(z) = z$. It is known that b is a fixed point for \mathcal{L}_λ (see section 3 in [27], or section 2 in [4]). From the uniqueness of the fixed point it follows the claim.

Therefore

$$b_{\lambda, A} = b(x) = \sup_{c \in \{1, 2, \dots, d\}^{\mathbb{N}}} S(x, c) = S(x, a(x)).$$

As the supremum of convex functions is convex we get that for λ fixed the function $b_{\lambda, A}$ varies in convex way with A .

Figure 1 suggests that $b(x)$ is obtained as $\sup\{S(x, w_1), S(x, w_2)\}$, where, w_1, w_2 , is in Σ . Later we will show that w_1, w_2 are in a periodic orbit of period 2 for the shift σ .

We point out that in a similar way the figure 1 in [35] suggests that $b(x)$ is obtained as $\sup\{S(x, w_i), i = 1, 2, 3, 4\}$, where, $w_i, i = 1, 2, 3, 4$,

is in Σ . Note that the potential A in that case is conjectured to have a maximizing probability in an orbit of period 4 (see [15]).

Note that the S is not periodic in x , but b is.

We define $\pi(x) = i$, if x is in the image of $\tau_i(S^1 - \{0, 1\})$, $i \in \{1, 2, \dots, d\}$.

Note also [35] that

$$S(T(x), \pi(x)a) = A(x) + \lambda S(x, a).$$

Or, in another way, for any $a = (a_0, a_1, \dots)$

$$S(x, a) = A(\tau_{a_0}(x)) + \lambda S(\tau_{a_0}(x), \sigma(a)).$$

In this way we get that $\phi(x, a) = (x, S(x, a))$ is a change of coordinates from F to $\Theta(x, a) = (T(x), \pi(x)a)$ [35].

$\Theta(x, a)$ is forward invariant in the upper boundary \mathcal{B} of the attractor

Note also that $\Theta^{-1}(x, a) = (\tau_{a_0}(x), \sigma(a))$ (when defined).

Then we can also write (in the case is well defined)

$$S(x, a) = A(\tau_{a_0}(x)) + \lambda S(\tau_{a_0}(x), \sigma(a)).$$

Definition 6. Consider a fixed \bar{x} and define

$$W(x, a) = S(x, a) - S(\bar{x}, a).$$

We call such W the λ -involution kernel for A .

then,

$$\begin{aligned} \lambda W(\tau_{a_0}(x), \sigma(a)) - W(x, a) &= \\ [\lambda S(\tau_{a_0}(x), \sigma(a)) - \lambda S(\bar{x}, \sigma(a))] - [S(x, a) - S(\bar{x}, a)] &= \\ [\lambda S(\tau_{a_0}(x), \sigma(a)) - S(x, a)] - [\lambda S(\bar{x}, \sigma(a)) - S(\bar{x}, a)] &= \\ -A(\tau_{a_0}(x)) + [\lambda S(\bar{x}, \sigma(a)) - S(\bar{x}, a)]. \end{aligned}$$

Definition 7. If we denote

$$A^*(a) = [\lambda S(\bar{x}, \sigma(a)) - S(\bar{x}, a)],$$

we get the λ -coboundary equation: for any (x, a)

$$A^*(a) = A(\tau_{a_0}(x)) + [\lambda W(\tau_{a_0}(x), \sigma(a)) - W(x, a)].$$

This kind of result is similar to the ones in [3] and [30] (but, with an extra parameter λ)

Note that W depends on the \bar{x} we choose. Therefore, $A^* = A_x^*$ depends of the \bar{x} .

If we consider another base point x_1 instead \bar{x} , in order to get a different $W_1(x, a) = S(x, a) - S(x_1, a)$, then one can show that the corresponding A_1^* (to A and W_1) satisfies

$$A_1^* = A^* + \lambda(g \circ \sigma) - g,$$

for some continuous g . Note that $W - W_1$ just depends on a .

For the dual problem it will be necessary to consider the following problem: finding a function $d^* = d_\lambda^*$ which satisfies for all a

$$\lambda d^*(a) = \max_{\sigma(c)=a} \{d^*(c) + A^*(c)\}.$$

In fact one can do more, it is possible to find a continuous function d^* that solves

$$\lambda d^*(\sigma(c)) = d^*(c) + A^*(c), \forall c \in \Sigma.$$

Just take, as in [2],

$$\begin{aligned} d^*(c) &= - \sum_{j=0}^{\infty} \lambda^j A^*(\sigma^j(c)) = \\ &- \sum_{j=0}^{\infty} \lambda^j [\lambda S(\bar{x}, \sigma^{j+1}(c)) - S(\bar{x}, \sigma^j(c))] = -S(\bar{x}, c). \end{aligned}$$

In this case the corresponding rate function in the dual problem $R^*(c) = \lambda d^*(\sigma(c)) - d^*(c) - A^*(c)$ is constant equal zero. This situation is quite different from the analogous dual problem in [30].

Definition 8. We call d_λ^* **the dual λ -calibrated subaction**.

We assume, without loss of generality, that $A > 0$. Then, $V = b > 0$.

It is natural to consider the sum $\sum R^*(\sigma^n)(z)$ in the dual problem (see [3] [30] [13]), but now this sum is zero.

In order to have a notation similar to [30] we denote V and V^* , respectively, the λ -calibrated for A and A^* .

Note that for all (x, a)

$$\begin{aligned} (V^* + V - W)(x, a) &= -S(\bar{x}, a) + b(x) + S(\bar{x}, a) - S(x, a) = \\ &b(x) - S(x, a) \geq 0. \end{aligned}$$

If a is a realizer for x , then $(V^* + V - W)(x, a) = 0$.

Given A (and, a certain choice of A^* and W) the next result claims that the dual of R is R^* , and the corresponding involution kernel is $(V^* + V - W)$.

Proposition 9.

$$R(\tau_w x) = (V^* + V - W)(x, w) - \lambda(V^* + V - W)(\tau_w x, \sigma(w)).$$

Proof: We know that

$$\lambda V^*(\sigma(w)) - V^*(w) = A^*(w),$$

and, now using $x = T(\tau_w x)$, we get

$$\begin{aligned} V(x) - \lambda V(\tau_w x) &= V(T(\tau_w x)) - \lambda V(\tau_w x) = \\ &= -A(\tau_w x) + A(\tau_w x) = R(\tau_w x) + A(\tau_w x). \end{aligned}$$

Substituting the above in the previous equation we get

$$\begin{aligned} (V^* + V - W)(x, w) - \lambda (V^* + V - W)(\tau_w(x), \sigma(w)) &= \\ [V^*(w) - \lambda V^*(\sigma(w))] + [V(x) - \lambda V(\tau_w x)] - W(x, w) + \lambda W(\tau_w x, \sigma(w)) &= \\ -A^*(w) + R(\tau_w(x)) + A(\tau_w(x)) + \lambda W(\tau_w(x), \sigma(w)) - W(x, w) &= R(\tau_w(x)), \end{aligned}$$

because $A^*(w) = A(\tau_w x) + \lambda W(\tau_w x, \sigma(w)) - W(x, w)$. So the claim follows. \square

We present now a brief outline of Transport Theory (see [36] [37] as a general reference).

Definition 10. We denote by $\mathcal{K}(\mu, \mu^*)$ the set of probabilities $\hat{\eta}(x, w)$ on $\hat{\Sigma} = S^1 \times \Sigma$, such that

$$\pi_x^*(\hat{\eta}) = \mu, \text{ and } \pi_w^*(\hat{\eta}) = \mu^*.$$

The Classical Transport Theory is not a Dynamical Theory. One have to consider a dynamical defined cost in order to get a relation of the optimal plan with the iteration of a Dynamical System.

We are going to consider bellow the cost function $c(x, w) = -W(x, w) = -W_\lambda(x, w)$ for a λ -involution kernel of the Holder potential A .

The Kantorovich Transport Problem: consider the minimization problem

$$C(\mu, \mu^*) = \inf_{\hat{\eta} \in \mathcal{K}(\mu, \mu^*)} \int \int -W(x, w) d\hat{\eta}.$$

Definition 11. A probability $\hat{\eta}$ on $\hat{\Sigma}$ which attains such infimum is called an optimal transport probability, or, an optimal plan, for $c = -W$.

We will show later that for μ_λ and μ_λ^* there exists and $\hat{\sigma}$ invariant probability $\hat{\mu}_{min}$, which attains the optimal transport cost.

Definition 12. A pair of continuous functions $f(x)$ and $f^\#(w)$ will be called c -admissible (or, just admissible for short) if

$$f^\#(w) = \min_{x \in \Sigma} \{-f(x) + c(x, w)\}.$$

We denote by \mathcal{F} the set of admissible pairs.

The Kantorovich dual Problem: given the cost $c(x, w)$ consider the maximization problem

$$D(\mu, \mu^*) = \max_{(f, f^\#) \in \mathcal{F}} \left(\int f d\mu + \int f^\# d\mu^* \right).$$

Definition 13. A pair of admissible $(f, f^\#) \in \mathcal{F}$ which attains the maximum value will be called an optimal Kantorovich pair.

Under quite general conditions [36] $D(\mu, \mu^*) = C(\mu, \mu^*)$.

We denote

$$\Gamma = \Gamma_V = \{(x, w) \in S^1 \times \Sigma \mid V(x) = (-V^* + W)(x, w)\}.$$

A Classical Theorem [36]: if $\hat{\eta}$ is a probability in $\mathcal{K}(\mu, \mu^*)$, $(f, f^\#)$ is an admissible pair, and the support of $\hat{\eta}$ is contained in the set

$$\{(x, w) \in \hat{\Sigma} \mid \text{such that } (f(x) + f^\#(w)) = c(x, w)\},$$

then, $\hat{\eta}$ in an optimal plan for c and $(f, f^\#)$ is an optimal pair in \mathcal{F} .

This is the so called slackness condition of Linear Programming (see [37] Remark 5.13 page 59).

We will show that for the problem $D(\mu_\lambda, \mu_\lambda^*)$ the functions $-b$ and $-d^*$ define an optimal Kantorovich pair. From this becomes clear the importance of the set Γ .

We claim first that $-V = -b$ and $-V^* = -d^*$ are $-W$ -admissible. Indeed,

$$p(x, w) := (V^* + V - W)(x, w) \geq 0.$$

Moreover, for each x there exists a w which is a realizer and then $p(x, w) = 0$.

Therefore, for each x we have that

$$V(x) = \max_{w \in \Sigma} \{-V^*(w) + W(x, w)\} = \max_{w \in \Sigma} S(x, w).$$

We can say that V is the W transform of $-V^*$ [36] [37].

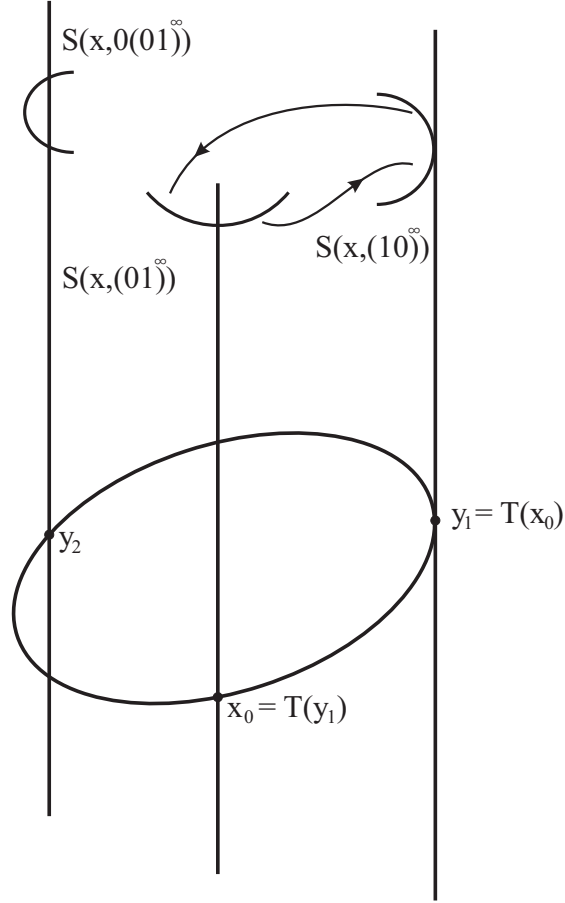


Fig 3).

Note that

$$\Gamma = \{(x, w) \in S^1 \times \Sigma \mid p(x, w) = 0\}.$$

We will show that the infimum of the cost $-W$, denoted $c(A, \lambda)$, is equal to $\int -V^* d\mu_\lambda^* + \int -V d\mu_\lambda$.

The next proposition is similar to a result on [30].

Proposition 14. (Fundamental relation): *for any (x, w)*

$$R(\tau_w x) = p(x, w) - \lambda p(\tau_w x, \sigma(w)) \quad (FR1).$$

From this main equation we get: if $\mathbb{T}^{-1}(x, w) = (\tau_w x, \sigma(w))$, then

- a) $p - \lambda p \circ \mathbb{T}^{-1}(x, w) = R(\tau_w x) \geq 0$;*
- b) Γ_V is invariant by the action of \mathbb{T}^{-1} ;*
- c) if $a = (i_0, i_1, i_2, \dots)$ is optimal for x , then $\sigma^n(a)$ is optimal for $(\tau_{i_{n-1}} \circ \dots \circ \tau_{i_1} \circ \tau_{i_0})(x)$.*

Proof:

The first claim a) is a trivial consequence of the definition of \mathbb{T}^{-1} . The second one it is a consequence of: $p \geq 0$, and

$$p - \lambda (p \circ \mathbb{T}^{-1})(x, w) \geq 0$$

$$p(x, w) \geq \lambda (p \circ \mathbb{T}^{-1})(x, w).$$

From the above we get that in the case (x, w) is optimal, then, $\mathbb{T}^{-1}(x, w)$ is also optimal. Indeed, we have that

$$p(x, w) = 0 \rightarrow p(\tau_w(x), \sigma(w)) = 0.$$

Item c) follows by induction. □

In this way \mathbb{T}^{-n} spread optimal pairs. This is a nice property that has no counterpart in the Classical Transport Theory.

Take now $(z_0, w_0) \in \Gamma_V$ and, for each n ,

$$\hat{\mu}_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\mathbb{T}^{-j}(z_0, w_0)}$$

Note that $\mathbb{T}^{-j}(z_0, w_0)$ is optimal and its closure is in the optimal transport set (pairs of realizers).

We claim that any weak limit of convergent subsequence $\hat{\mu}_{n_k}$, $k \rightarrow \infty$, will define a probability $\hat{\mu}$ which is optimal for the transport problem for $-W$ and its marginals. In this way we will show the existence of a \mathbb{T} -invariant probability on $S^1 \times \Sigma$ which is optimal for the associated transport problem.

Indeed, we considered before a certain z_0 , its realizer w_0 , and then a convergent subsequence μ_{n_k} (notation of last section), $n_k \rightarrow \infty$, in order to get μ_λ . If we consider above the correspondig subsequence $\mathbb{T}^{-n_k}(z_0, w_0)$ we get that the projection of $\hat{\mu}$ on the S^1 coordinate is μ_λ .

In an analogous way, we consider as before a certain z_0 , its realizer w_0 , and then a convergent subsequence m_k to define μ_λ^* . If we consider above a subsequence m_k of the previous sequence n_k (last paragraph) we get that the projection of $\hat{\mu}$ we get on the Σ coordinate is μ_λ^* .

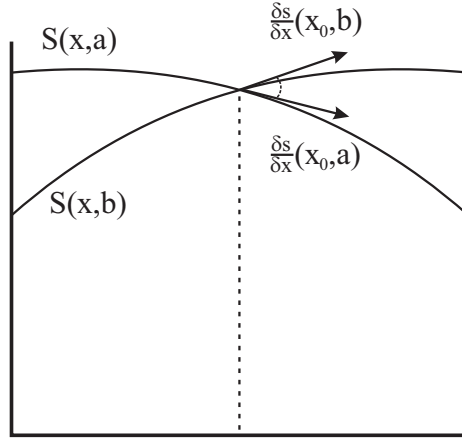


Fig 4).

Then any probability $\hat{\mu}$ obtained in this way is such that projects respectively on μ_λ and μ_λ^* , and, moreover, satisfies

$$\int -W d\hat{\mu} = \int (-V^*) d\mu^* + \int (-V) d\mu_\lambda.$$

Therefore, $c(A, \lambda) = \int (-d^*) d\mu^* + \int (-b) d\mu_\lambda$.

From the above we get the proof of our main result in this section:

Theorem 15. *For the probabilities $\mu_\lambda, \mu_\lambda^*$ and the cost $-W$, the associated transport problem is such that the functions $-b$ and $-d^*$ define an optimal Kantorovich pair, and, the optimal plan is invariant by \mathbb{T} .*

In [28] other kind of results in Ergodic Transport Theory are considered.

3. THE TURNING POINT

Remember that given x there exists a certain $a = a(x)$ which realizes x , and, also that the calibrated subcation b satisfies

$$b(x) = \sup_{c \in \{1, 2, \dots, d\}^{\mathbb{N}}} S(x, c) = S(x, a(x)).$$

Conjecture: For a generic A in class C^1 this $a(x)$ can be taken in locally constant way up to a finite number of points x .

We claim that it is necessary the above to be true, in order, the main conjecture of [6] to be true.

A turning point (see [30]) is a point in S^1 where there is a change of $a(x) \in \Sigma$. By this we mean there is no possible way of avoiding this by the choice of $a(x)$.

Conjecture: The unstable manifold by F of a periodic point breaks (became discontinuous) in the turning point.

We claim that if the above is true, then, the main conjecture of [6] is also true.

We are able to show that this is true under the twist condition in the generic case in the sense of the claim of Theorem 4.

We know that if A is C^1 , then for fixed a we have that $S(x, a)$ is C^1 in x .

As $F(x, s) = (T(x), \lambda s + A(x))$, given a point x and its preimages y_i , $i \in \{1, 2, \dots, d\}$, then $F(y_i, b(y_i)) = (x, b(x))$, if, and only if y_i is a maximum for $b(x) = \max_{T(y)=x} \{\lambda b(y) + A(y)\}$.

The upper boundary is the (upper part of the) boundary of the set of points which have a bounded set of pre-images.

We claim that if μ_λ is a probability with support on a T -periodic orbit of period k , and x_0 is a point on the support of μ_λ , then, by the λ -cohomological equation (an equality) we can take the corresponding realizer \bar{a} as a point in a σ -periodic orbit of period k . Indeed, suppose for simplification we consider $d = 2$ and an orbit of period 2, denoted by $\bar{x}, T(\bar{x})$. Consider $y_1 = \tau_1(T(\bar{x}))$ and $y_2 = \tau_2(T(\bar{x}))$. Suppose without loss of generality that y_1 (or, $1 \in \{1, 2\}$) realizes $T(\bar{x})$ and y_2 not. Then, $T(y_1) = \bar{x}$, otherwise, $b(T(\bar{x})) = b(y_2)$, and there points z (in the vertical fiber by y_2) close and above $b(y_2)$ which do not have a bounded set of pre-images. Using the order on the fiber and the fact that $b(y_2) < b(z) < b(T(\bar{x}))$ we reach a contradiction. In this way $\bar{x} = y_1$.

The case of the general d and any period k is similar.

The bottom line is: for a given $(x_0, b(x_0))$ in the boundary of the attractor, with x_0 a T -periodic point, we get that $b(x_0) = S(x_0, \bar{a}) = S(x_0, a(x_0))$, where \bar{a} is in σ -periodic orbit with the same period.

Proposition 16. *For any x sufficiently close to x_0 (in a T -periodic orbit) we have that $b(x) = S(x, \bar{a}) = S(x, a(x_0))$. This shows that the boundary of the attractor is a piece of differentiable curve when we consider points close by $(x_0, b(x_0))$.*

Proof: Suppose, for simplification, that the period $k = 2$.

Assume, without loss of generality, that $\tau_1(x_0)$ realizes $b(x_0)$.

Note that for a generic A , in the case x_0 is T -periodic, the function $a(x)$ is locally constant around x_0 . In other words, $\tau_1(x)$ realizes $b(x)$, for x close to x_0 .

Indeed, the function $F(x, s) = (T(x), \lambda s + A(x))$ is strictly monotone in the fiber x . Then, x has to realize $b(\tau_1(x))$. Then, from the reasoning describe just above (before the claim of the present proposition) we get that τ_2 is the symbol which realizes $b(\tau_1(x))$.

Therefore, for fixed period 2, $a(x) = (1, 2, 1, 2, 1, \dots)$, and $a(T(x)) = (2, 1, 2, 1, 2, \dots)$ for x close to x_0 . This is true, if we move x more far away from x_0 , until x reaches a turning point.

Finally, we use the fact that $b(x) = S(x, \bar{a})$ is differentiable on x .

The above is also true for an open set of potentials close to A . □

Consider x in a small neighbourhood of x_0 of T -period 2 as above.

Note that for such x we have that $F^2(x, b(x)) = (T^2(x), b(T^2(x)))$.

Indeed, $F^2(x, b(x)) = (T^2(x), \lambda^2 b(x) + \lambda A(x) + A(T(x)))$.

But, by the realizing property described above

$$b(T^2(x)) = \lambda b(T(x)) + A(T(x)) = \lambda(\lambda b(x) + A(x)) + A(T(x)).$$

Then, $F^2(x, b(x)) = (T^2(x), b(T^2(x)))$, and the piece of curve $(x, b(x))$ (in the upper boundary of the attractor) is invariant an one dimensional. It is a piece of unstable manifold for F in the point $(x_0, b(x_0))$ because locally $T(x)$ expand distances (see figure 3). This will be so, if we move x far away from x_0 , until one the points x , or, $T(x)$ reaches a turning point.

Suppose μ_λ is a probability with support on two T -fixed points y_1 and y_2 . Then, by the λ -cohomological equation we can take two points \bar{a}_1 and \bar{a}_2 which are also fixed and associated respectively to y_1 and y_2 by the above procedure. That is, $b(y_1) = S(y_1, \bar{a}_1)$ and $b(y_2) = S(y_2, \bar{a}_2)$.

4. TWIST PROPERTIES

In this section we want to analyzed some general examples, where the potential is twist, and, where the claim of the main conjecture can be shown to be true in full extent (not for just for λ large).

Let $A : S^1 \rightarrow \mathbb{R}$ be a C^1 or C^2 potential, $\lambda \in (0, 1)$, $a = (a_0, a_1, \dots) \in \{0, 1\}^{\mathbb{N}}$, and, τ_i , $i = 0, 1$, be the inverse branches of $T(x) = 2x \mod 1$.

We can also consider the same problem for $A : [0, 1] \rightarrow \mathbb{R}$ of class C^1 .

We also define $\tau_{k,a}x = \tau_{a_k} \circ \tau_{a_{k-1}} \dots \tau_{a_0}(x)$.

We consider the control $a = (a_0, a_1, \dots) \in \{0, 1\}^{\mathbb{N}}$ and the sum

$$S(x, a) = \sum_{k=0}^{\infty} \lambda^k A(\tau_{k,a}x).$$

We also denote $S_A(x, a)$.

We consider in $\{0, 1\}^{\mathbb{N}}$ the lexicographic order.

Definition 17. *We say that A satisfies the twist condition, if, for any $a < b$, we have*

$$\frac{\partial S}{\partial x}(x, a) - \frac{\partial S}{\partial x}(x, b) > 0.$$

The twist condition (also known as modular or Monge condition) is quite natural in applications (see [16] [7])

Remember that

$$W(x, a) = S(x, a) - S(\bar{x}, a).$$

In this way if A is of class C^1 , then, for fixed a the function $W(x, a)$ is differentiable on x .

We recall the following result from [30].

Proposition 18. *If A is twist, then $x \rightarrow a(x)$ is monotonous non-increasing, where for each x we denote by $a(x)$ a realizer of x .*

Therefore, as

$$b(x) = \max_{w \in \Sigma} \{-V^*(w) + W(x, w)\} = \max_{a \in \Sigma} S(x, a),$$

if, A satisfies the twist condition, and, there exists just a finite number of realizers $a(x)$, for the set of all possible x , then $b(x)$ is piecewise smooth.

The turning points are the ones where there exist a change of the realizer. If A is differentiable we can differentiate S with respect to x

In figure 4 we show the geometrical description of the analytical result we are looking for.

$$\frac{\partial S}{\partial x}(x, a) = \sum_{k=0}^{\infty} \lambda^k A'(\tau_{k,a}x) \frac{\partial}{\partial x} \tau_{k,a}x.$$

We observe that $\tau_{k,a}x$ has an explicit expression, $\tau_{k,a}x = \frac{1}{2^{k+1}}x +$

$$\psi_k(a), \text{ where } \psi_k(a) = \frac{1}{2^{k+1}} \sum_{j=0}^k \frac{a_j}{2^j}.$$

Thus,

$$\frac{\partial S}{\partial x}(x, a) = \sum_{k=0}^{\infty} \lambda^k A'(\tau_{k,a}x) \frac{1}{2^{k+1}} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\lambda}{2}\right)^k A'(\tau_{k,a}x).$$

We will estimate now $\frac{\partial S}{\partial x}(x, a)$ for $A(x) = x^m$ for $m \leq 2$.

a) $A(x) = c$

In that case, $S(x, a) = \sum_{k=0}^{\infty} \lambda^k c = \frac{c}{1-\lambda}$, so $\frac{\partial S}{\partial x}(x, a) = 0$.

b) $A(x) = x$

In that case, $S(x, a) = \sum_{k=0}^{\infty} \lambda^k \tau_{k,a}x$, so $\frac{\partial S}{\partial x}(x, a) = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\lambda}{2}\right)^k = \frac{1}{2-\lambda}$.

c) $A(x) = x^2$

In that case, $S(x, a) = \sum_{k=0}^{\infty} \lambda^k (\tau_{k,a}x)^2$, so $\frac{\partial S}{\partial x}(x, a) = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\lambda}{2}\right)^k 2(\tau_{k,a}x) = \sum_{k=0}^{\infty} \left(\frac{\lambda}{2}\right)^k (\tau_{k,a}x)$.

Thus,

$$\frac{\partial S}{\partial x}(x, a) = \sum_{k=0}^{\infty} \left(\frac{\lambda}{2}\right)^k \left[\frac{1}{2^{k+1}}x + \psi_k(a) \right] = \frac{2}{4-\lambda}x + \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\lambda}{4}\right)^k \theta_k(a),$$

where $\lim_{k \rightarrow \infty} \theta_k(a) = \lim_{k \rightarrow \infty} \sum_{j=0}^k \frac{a_j}{2^j} \rightarrow y(a) \in [0, 1]$; and, $y(a)$ is the real number with binary expansion a .

From the recursiveness $\theta_{k+1}(a) = \theta_k(a) + \frac{a_{k+1}}{2^{k+1}}$, we get,

$$\begin{aligned} H &= \sum_{k=0}^{\infty} \left(\frac{\lambda}{4}\right)^k \theta_k(a) = \theta_0(a) + \sum_{k=0}^{\infty} \left(\frac{\lambda}{4}\right)^{k+1} \theta_{k+1}(a) \\ &= \frac{\lambda}{4} H + Z(a). \end{aligned}$$

Thus, $H = \frac{4}{4-\lambda} Z(a)$, where $Z(a) = \sum_{k=0}^{\infty} \left(\frac{8}{\lambda}\right)^k a_k$ is the expansion in the base $\frac{8}{\lambda}$.

In order to estimate $\frac{\partial S}{\partial x}(x, a)$ we can derive

$$\frac{\partial S}{\partial x}(x, a) = \frac{2}{4 - \lambda}x + \frac{2}{4 - \lambda}Z(a)$$

We use the notation $S_A(x, a) = \sum_{k=0}^{\infty} \lambda^k A(\tau_{k,a}x)$, to show that S depends linearly of A . So, if we consider $A(x) = c_0 + c_1x + c_2x^2$, then, we get

$$S_A(x, a) = c_0 + c_1S_x + c_2S_{x^2}.$$

We can also compute $\frac{\partial S}{\partial x}(x, a)$,

$$\frac{\partial S}{\partial x}(x, a) = c_0 \cdot 0 + c_1 \frac{1}{2 - \lambda} + c_2 \left(\frac{2}{4 - \lambda}x + \frac{2}{4 - \lambda}Z(a) \right),$$

or,

$$\frac{\partial S}{\partial x}(x, a) = \left(\frac{c_1}{2 - \lambda} + \frac{2c_2}{4 - \lambda}x \right) + \frac{2c_2}{4 - \lambda}Z(a).$$

We are now interested in transversality conditions.

Let and $a \neq b \in \{0, 1\}^{\mathbb{N}}$ consider $S(x, a), S(x, b)$ as above. We say that $S(x, a) \pitchfork S(x, b)$ in x , if

- a) $S(x, a) \neq S(x, b)$, or,
- b) $S(x, a) = S(x, b)$ and $\frac{\partial S}{\partial x}(x, a) \neq \frac{\partial S}{\partial x}(x, b)$.

Lemma 19. *If A is quadratic and $a, b \in \{0, 1\}^{\mathbb{N}}$ is such that $a \neq b$, then, $Z(a) \neq Z(b)$ and $S(x, a) \pitchfork S(x, b), \forall x$. In particular, $\{x | S(x, a) = S(x, b)\}$ is a single point.*

Proof. Suppose $A = c_0 + c_1S_x + c_2S_{x^2}$ and $S(x, a) = S(x, b)$, then,

$$\frac{\partial S}{\partial x}(x, a) - \frac{\partial S}{\partial x}(x, b) = \frac{2c_2}{4 - \lambda}(Z(a) - Z(b)).$$

In order to see that $\{x | S(x, a) = S(x, b)\}$ is a single point we just observe that $Z(\cdot)$ is a monotonous representation of $\{0, 1\}^{\mathbb{N}}$. Therefore, we claim that the angle between $S(x, a)$ and $S(x, b)$ has always the same orientation (but it depends on the signal of c_2 off course). To see that, we notice that $Z(a) \neq Z(b)$, provided that $a \neq b$, because $\sum_{k=N+1}^{\infty} (\frac{\lambda}{8})^k a_k \leq (\frac{\lambda}{8})^N \frac{\frac{\lambda}{8}}{1 - \frac{\lambda}{8}} = (\frac{\lambda}{8})^N \frac{\lambda}{8 - \lambda} < (\frac{\lambda}{8})^N$, provided that $\frac{\lambda}{8} < 1$ and $a_k \leq 1$, so two distinct sequences can not represents the same real number. \square

From this follows that $S(x, a)$ satisfies the twist condition. The same property is true for the λ -involution kernel.

This property is stable in the set of quadratic potentials of this form.

From the above it follows the main result of the paper which shows the above mentioned conjecture in the case A is quadratic.

Theorem 20. *Let $A(x) = (x - \frac{1}{2})^2$ continuous in S^1 , in this case $c_2 = 1$, and,*

$$\frac{\partial S}{\partial x}(x, a) - \frac{\partial S}{\partial x}(x, b) = \frac{2}{4 - \lambda}(Z(a) - Z(b)).$$

Note that when $\lambda \rightarrow 1$ the angles remain bounded away from zero.

Therefore, in this case the boundary of the attractor, for $\lambda < 1$, is the union of two unstable manifolds of F -periodic points of period two. It is a finite union of pieces of unstable manifolds.

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Instituto de Matemática
UFRGS
Av. Bento Goncalves, 9500
Porto Alegre - 91.500 Brazil